Brewster Angle and Vanishing Polarization of Wave Reflected by Conductor-Backed Water Slab

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***Abstract*—For ease of visualization of Brewster angle, we set up a simple experiment with copper-backed water slab to show near-full transmission of parallel polarization. Using a circularly polarized microstrip patch antenna connected to a portable USB signal generator as the transmitter and a linearly polarized patch with a portable USB spectrum analyzer as the receiver, the low parallel polarization reflection level is observable at the desired angle, calculated with undergraduate-level electromagnetic theory. With the help of the experiment, the effect is intuitively understood and easily reproduced by students in classrooms.**

***Index Terms*—Brewster angle, polarization, electromagnetics, education.**

I. Introduction

Brewster angle, the angle at which electromagnetic wave incidents another dielectric surface so that the parallel polarization component completely transmits without reflection, leaving the reflected wave fully polarized, is taught in undergraduate electromagnetics [1], but the concept is often hard for students to grasp. Here, we provide a simple experiment with easy setup to assist classroom learning of Brewster angle. Theoretically, the incident wave enters an infinite half space. In experiments, it is difficult to have an enormous slab of dielectric just to demonstrate Brewster effect. If we place a perfect electric conductor (PEC) in the back of a dielectric slab of moderate thickness, we can derive the new Brewster angle formula, and with careful selection of material, the experiment is feasible and easy to set up for both instructor demonstration and hands-on student participation.

In this work, we use water backed by a large copper sheet as the dielectric with our wave source emitting at a fixed location. From our analytical calculations based on undergraduate-level electromagnetic theory [1], the reflection coefficients ($R_{\parallel}$ and $R_{\perp}$) for incident waves on PEC-backed dielectric slab are controlled by three variables: material properties (μ, ε, σ), material thickness (d) and incident angle (θ). Our choice of material depends on the ease of access, size, and the feasibility of the incident angle, which leads us to use room temperature (25°) water.

The antennas used in this experiment also follow the rule of simplicity. Implementing the operational principles of microstrip patch antenna and quadrature hybrid [2], we design antennas at 2.45 GHz on simple 1.6-mm FR4 boards: a linearly polarized (LP) rectangular patch antenna and a single-feed circularly polarized (CP) square patch antenna with polarizations that are switchable by interchanging the termination load (50 Ω) with the feeding cable as shown in Fig. 1. The transmitting (Tx) CP patch is connected to Signal Hound's USB signal generator USB-TG44A while the receiving (Rx) LP patch is connected to their spectrum analyzer USB-SA44B. For the PEC-backed water, a kiddie pool is used as a container with a large copper sheet horizontally placed at the bottom and another one standing vertically, elevated by 12 cm from the base to block potential direct air transmission between the Tx and Rx antennas. By adding water into the pool and observing the signal received at Rx, we can see that near-total transmission of parallel polarized wave occurs at a particular water depth, that is, when the incident angle reaches the desired angle, and hence the vanishing polarization in the reflected wave.

II. Formulation

The step-by-step derivation of the reflection coefficients of parallel and perpendicular polarizations are given in the appendix. Therefore, with the angle of incidence θ, we have

$$\begin{cases} R_{\parallel} = \dfrac{\mu_1 k_{x_0}\cos k_{x_1}d - j\varepsilon_0\eta_1^2 k_{x_1}\sin k_{x_1}d}{\mu_1 k_{x_0}\cos k_{x_1}d + j\varepsilon_0\eta_1^2 k_{x_1}\sin k_{x_1}d}e^{2jk_{x_0}d} & \text{(1a)} \\ R_{\perp} = \dfrac{j\mu_1 k_{x_0}\sin k_{x_1}d - \mu_0 k_{x_1}\cos k_{x_1}d}{j\mu_1 k_{x_0}\sin k_{x_1}d + \mu_0 k_{x_1}\cos k_{x_1}d}e^{2jk_{x_0}d}, & \text{(1b)} \end{cases}$$

where $k_{x_0} = k_0\cos\theta$, $k_{x_1} = \omega\sqrt{\mu_1^2/\eta_1^2 - \mu_0\varepsilon_0\sin^2\theta}$, and intrinsic impedance $\eta_1 = \sqrt{j\omega\mu_1/(\sigma + j\omega\varepsilon_1)}$.

Here, we hope to realize the full transmission of parallel polarization wave, i.e., $R_{\parallel} = 0$. Without the PEC board, we know from [1, eq. (8.70a, 75a, 81)] that

$$\begin{cases} R'_{\parallel} = \dfrac{\eta_1\cos\phi - \eta_0\cos\theta}{\eta_1\cos\phi + \eta_0\cos\theta} & \text{(2a)} \\ R'_{\perp} = \dfrac{\eta_1\cos\theta - \eta_0\cos\phi}{\eta_1\cos\theta + \eta_0\cos\phi} & \text{(2b)} \end{cases}$$

and the Brewster angle is $\theta_B = \arctan\sqrt{\varepsilon_1/\varepsilon_0}$. Taking a closer look at the numerator of (2b) with the assumption of $\mu_1 = \mu_0$, $\varepsilon_1 > \varepsilon_0$, and $\sigma = 0$, we can see that $\cos\phi > \cos\theta$, and therefore, $R_{\perp}$ is never 0 when the dielectric is not air.

When we reintroduce the PEC board back into the setup, we know that there will be reflection at the PEC. If the angle of incidence is the original Brewster angle θ_B, the angle of refraction is $\phi = \arctan\sqrt{\varepsilon_0/\varepsilon_1}$. Since reflection mirrors wave path, the reflected wave now incidents the interface

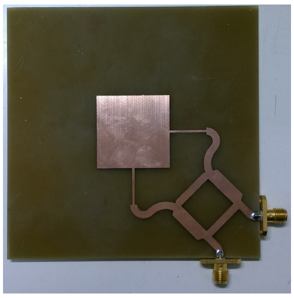

Fig. 1. Designed CP antenna.

from dielectric to air at ϕ. If the dielectric is lossless, all the refracted wave in the dielectric will refract back into the air. However, in a lossy dielectric, we can see a very low level of reflected parallel-to-perpendicularly polarized wave ratio, an effect closely resembling Brewster effect, when the wave incidents at a desired angle θ that is not necessarily θ_B.

With the parameters of different dielectric materials, we use a simple program we have written to help us find the desired incident angle θ, where the reflection of parallel polarized wave is small ($\left|R_{\parallel}\right| / \left|R_{\perp}\right| < -20$ dB) while keeping the dielectric thickness reasonable ($d < 0.10$ m). The most suitable material is water at room temperature (25°C), where, if we use $f = 2.45$ GHz, we have $\varepsilon(f) = 77.141 - 9.166i$, i.e., $\varepsilon_1 = 77.141\varepsilon_0$ and $\tan\delta = \sigma/\omega\varepsilon_1 = 0.1188$ [3]. At an incident angle of $\theta = 66°$ and dielectric thickness of $d = 1.05$ cm, the theoretical values of $\left|R_{\parallel}\right|$ and $\left|R_{\perp}\right|$ are –33.99 dB and –2.95 dB, respectively.

III. Experiment Setup

The purpose of this paper is to use a simple experiment that is doable in any classroom setting to demonstrate the low reflection of parallel polarized wave. Our top priority regarding the choice of dielectric material is its accessibility: the material should be easy to acquire, and not too bulky, lest difficult to bring into classrooms. Using our program, we find that room temperature water satisfies all. In addition, all resources we have used in this experiment are relatively cheap in the world of radio frequency, and also suitable for reproducing in lab sessions for students learning electromagnetics.

Depicted in Figs. 2 and 3, we set up a collapsible kiddie pool on a smooth surface (e.g. floor or leveled tables) and place a sleek copper board on the bottom as the reflective PEC. Since the desired θ is 66°, a substantial portion of the Tx wave will directly transmit to the Rx antenna. To avoid this situation, we place another copper board, this time vertically, 12 cm ($\approx \lambda$) above floor of the pool for blockage of direct transmission, still allowing space for reflection on water surface. Without the shielding copper at the water depth where near-full transmission of parallel polarization occurs, the reception level is 21.95 dB higher than that with the copper. It shows that the signals received without shielding is not from refection, but rather direct transmission. The need for the vertical copper sheet is then evident.

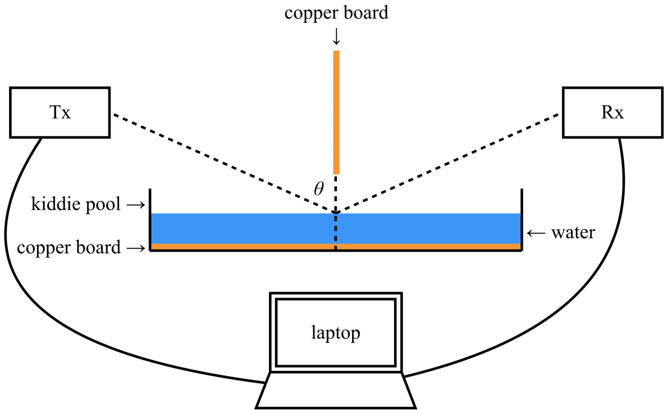

Fig. 2. Diagram for experiment setup.

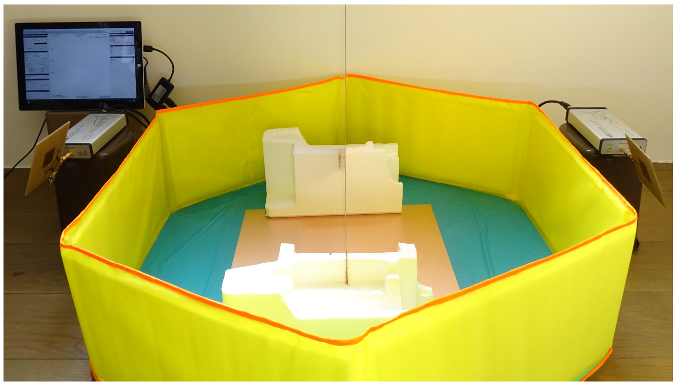

Fig. 3. Photo of experiment setup.

At the edge of the pool, we set up a CP patch antenna as the Tx antenna and an LP patch antenna as the Rx antenna 20 cm above ground, with both normal direction pointing at the center of the reflecting copper board at $\theta = 66°$. The distance between Tx and Rx antennas is roughly 90 cm. The CP and LP antennas are connected to a Windows laptop through Signal Hound's signal generator USB-TG44A and spectrum analyzer USB-SA44B respectively.

We will provide a short demonstration video and its YouTube link at the conference.

IV. Results

For all the following cases, left hand CP (LHCP) patch is used as the Tx antenna. Prior to adding water, the reception levels ($|S_{21}|$) of parallel and perpendicularly polarized wave are -39.32 dB and -36.26 dB respectively. The difference is a result of misalignment since we do not move neither the signal generator nor the spectrum analyzer despite the replacing of a 3.5 mm male-to-male straight-body adapter with a right-angle one at the Rx antenna to change the polarization from perpendicular to parallel. Relative to the phenomenon we are about to observe, their levels are comparable.

Then, we add fresh water into the pool. Dividing the measured parallel polarization level by that of perpendicular one, the result ($|S_{21\parallel}|/|S_{21\perp}|$) for water depth d from 0.90 cm to 1.20 cm is shown in Fig. 4. It is clear that near-full transmission of parallel polarization ($|S_{21\parallel}|/|S_{21\perp}| = -21.92$ dB) occurs when $d = 1.05$ cm. Thus, the vanishing polarization is observed.

V. Conclusion

The simple experiment design provided in this paper uses affordable and portable equipment, and can be integrated into laboratory curricula for electromagnetics, offering a new approach for visualization of Brewster effect for students.

Appendix

Under typical circumstances, especially in classrooms, we want to transmit wave from air (μ_0, ε_0) into some chosen dielectric (μ_1, ε_1, σ). Here, we will put the dielectric between $x = 0$ and $x = d$, where $x > d$ is air, and the PEC board is on the y axis as shown in Fig. 5a and 5b. The electromagnetic wave obliquely incidents the interface at an angle of θ at $(d, 0)$, also depicted in Fig. 5a and 5b.

Since there are no free charges or field sources, we know from Maxwell's equations that

$$\begin{cases} \nabla \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} & \text{(3a)} \\ \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \dfrac{\partial \vec{E}}{\partial t} & \text{(3b)} \\ \nabla \cdot \vec{E} = 0 & \text{(3c)} \\ \nabla \cdot \vec{H} = 0. & \text{(3d)} \end{cases}$$

We also know that the curl of curl of an arbitrary vector $\vec{V}$ is $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$. Apply the curl operator to Maxwell's equations, and we get the wave equations

$$\begin{cases} \nabla^2 \vec{E} - \varepsilon\mu \dfrac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \dfrac{\partial \vec{E}}{\partial t} = 0 & \text{(4a)} \\ \nabla^2 \vec{H} - \varepsilon\mu \dfrac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \dfrac{\partial \vec{H}}{\partial t} = 0. & \text{(4b)} \end{cases}$$

For time-harmonic field and complex notation, we have the Helmholtz equations [4, eq. (21.7, 8)]

$$\begin{cases} \nabla^2 \vec{E} + \left(\omega^2 \varepsilon\mu - j\omega\mu\sigma\right) \vec{E} = 0 & \text{(5a)} \\ \nabla^2 \vec{H} + \left(\omega^2 \varepsilon\mu - j\omega\mu\sigma\right) \vec{H} = 0. & \text{(5b)} \end{cases}$$

Using the intrinsic impedance $\eta = \sqrt{j\omega\mu/(\sigma + j\omega\varepsilon)}$, (5a) and (5b) become

$$\begin{cases} \nabla^2 \vec{E} + \dfrac{\omega^2 \mu^2}{\eta^2} \vec{E} = 0 & \text{(6a)} \\ \nabla^2 \vec{H} + \dfrac{\omega^2 \mu^2}{\eta^2} \vec{H} = 0. & \text{(6b)} \end{cases}$$

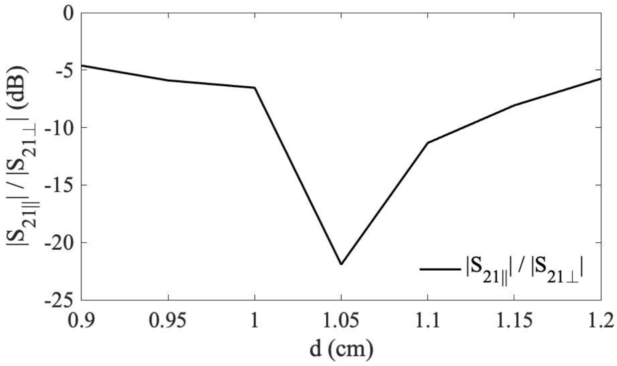

Fig. 4. Measured $|S_{21\parallel}|/|S_{21\perp}|$ with respect to water depth d.

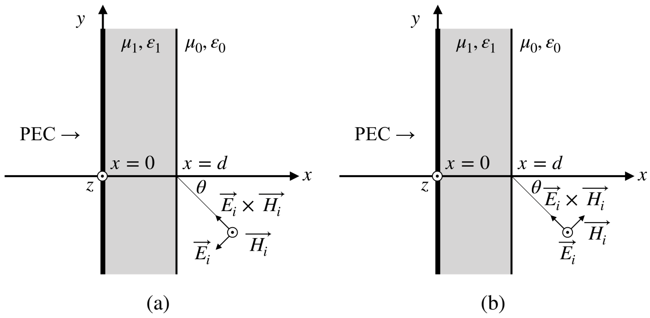

Fig. 5. Diagram for the cases of (a) parallel and (b) perpendicular polarization wave incidence.

A. Parallel Polarization

Here, we discuss parallel polarization, the polarization which the E field is parallel to the plane of incidence, first, as shown in Fig. 5a. Let us denote the H-field component of the parallel polarized incident wave (vertical polarization) as $\vec{H_i} = H_{z_i} \vec{a_z}$, where

$$H_{z_i} = A_0 e^{jk_0(x\cos\theta - y\sin\theta)} \tag{7}$$

and $k_0 = \omega\sqrt{\mu_0 \varepsilon_0}$ is the free space wave number. Also, we can see that the total H field in the air takes the form of

$$H_{z_0} = \left(A_0 e^{jk_0 x\cos\theta} + B_0 e^{-jk_0 x\cos\theta}\right) e^{-jk_0 y\sin\theta} \tag{8}$$

with B_0 as the reflected and refracted magnitudes of H field in the air. From (6b), we know that the total H field in the dielectric, H_{z_1}, satisfies

$$\frac{\partial^2 H_{z_1}}{\partial x^2} + \frac{\partial^2 H_{z_1}}{\partial y^2} + \frac{\omega^2 \mu_1^2}{\eta_1^2} H_{z_1} = 0, \tag{9}$$

where $\eta_1 = \sqrt{j\omega\mu_1/(\sigma + j\omega\varepsilon_1)}$. Using separation of variables for (9), we suppose $XY = H_{z_1}$, $X''Y = \partial^2 H_{z_1}/\partial x^2$, and $XY'' = \partial^2 H_{z_1}/\partial y^2$. Rewriting (9), the equation becomes

$$X''Y + XY'' + \frac{\omega^2\mu^2}{\eta^2} XY = 0$$
$$\Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} - \frac{\omega^2\mu^2}{\eta^2} = \Lambda,$$

and $Y = C_1 e^{\sqrt{\Lambda} y}$, where Λ and C_1 are arbitrary numbers. Since the boundary condition $H_{z_0}|_{x=d_+} = H_{z_1}|_{x=d_-}$ must be satisfied, we know from (8) that $\partial^2 H_{z_0}/\partial y^2\big|_{x=d_+} = -k_0 \sin^2\theta H_{z_0}$, and $\sqrt{\Lambda}$ must be $-jk_0 \sin\theta$. So we have

$$\frac{\partial^2 H_{z_1}}{\partial y^2} = -k_0^2 \sin^2\theta H_{z_1}. \tag{10}$$

Replacing (10) into (9), we get

$$\frac{\partial^2 H_{z_1}}{\partial x^2} = -\omega^2 \left(\frac{\mu_1^2}{\eta_1^2} - \mu_0 \varepsilon_0 \sin^2\theta \right) H_{z_1}. \tag{11}$$

Therefore, the H fields in the air ($\vec{H_0} = H_{z_0} \vec{a_z}$) and dielectric ($\vec{H_1} = H_{z_1} \vec{a_z}$) are described as

$$\begin{cases} H_{z_0} = \left(A_0 e^{jk_{x_0} x} + B_0 e^{-jk_{x_0} x} \right) e^{-jk_0 y \sin\theta} \\ H_{z_1} = \left(A_1 e^{jk_{x_1} x} + B_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta}, \end{cases} \tag{12}$$

where $k_{x_0} = k_0 \cos\theta$, and $k_{x_1} = \omega \sqrt{\mu_1^2/\eta_1^2 - \mu_0 \varepsilon_0 \sin^2\theta}$.

Now that we have the magnetic fields, we can derive the electric fields from (3b) and get

$$\begin{cases} E_{x_1} = -\frac{k_0 \sin\theta}{\omega \mu_1} \eta_1^2 \left(A_1 e^{jk_{x_1} x} + B_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta} \\ E_{y_1} = -\frac{k_{x_1}}{\omega \mu_1} \eta_1^2 \left(A_1 e^{jk_{x_1} x} - B_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta}, \end{cases} \tag{13}$$

where $\vec{E_1} = E_{x_1} \vec{a_x} + E_{y_1} \vec{a_y}$ is the E field in dielectric and $\vec{E_0} = E_{x_0} \vec{a_x} + E_{y_0} \vec{a_y}$ is for air by replacing all subscript 1's with 0's.

Since we have a PEC at $x = 0$, $E_{y_1}|_{x=0_+} = 0$. Then from $A_1 e^{jk_{x_1} d}|_{x=0_+} = B_1 e^{-jk_{x_1} d}|_{x=0_+}$, we can see that $B_1 = A_1$. Applying the boundary condition $E_{y_0}|_{x=d_+} = E_{y_1}|_{x=d_-}$ at $x = d$ and with the reflection coefficient $R_{\|} = B_0/A_0$ at the interface of air and dielectric, we have

$$\begin{aligned} &\frac{k_{x_0} A_0}{\varepsilon_0} \left(e^{jk_{x_0} d} - R_{\|} e^{-jk_{x_0} d} \right) \\ =&\frac{k_{x_1} A_1}{\mu_1} \eta_1^2 \left(e^{jk_{x_1} d} - e^{-jk_{x_1} d} \right). \end{aligned} \tag{14}$$

Also, from $H_{z_0}|_{x=d_+} = H_{z_1}|_{x=d_-}$, we have

$$A_0 \left(e^{jk_{x_0} d} + R_{\|} e^{-jk_{x_0} d} \right) = A_1 \left(e^{jk_{x_1} d} + e^{-jk_{x_1} d} \right), \tag{15}$$

and thus

$$A_1 = \frac{A_0 \left(e^{jk_{x_0} d} + R_{\|} e^{-jk_{x_0} d} \right)}{e^{jk_{x_1} d} + e^{-jk_{x_1} d}}. \tag{16}$$

Replacing (16) into (14), we can solve

$$R_{\|} = \frac{\mu_1 k_{x_0} \cos k_{x_1} d - j\varepsilon_0 \eta_1^2 k_{x_1} \sin k_{x_1} d}{\mu_1 k_{x_0} \cos k_{x_1} d + j\varepsilon_0 \eta_1^2 k_{x_1} \sin k_{x_1} d} e^{2jk_{x_0} d}. \tag{17}$$

B. Perpendicular Polarization

Following similar steps as Section A with Fig. 5b, we have the E-field component of the perpendicularly polarized incident wave (horizontal polarization) $\vec{E_i} = E_{z_i} \vec{a_z}$, where

$$E_{z_i} = C_0 e^{jk_0 (x \cos\theta - y \sin\theta)}. \tag{18}$$

Using (6a), we eventually have

$$\begin{cases} E_{z_0} = \left(C_0 e^{jk_{x_0} x} + D_0 e^{-jk_{x_0} x} \right) e^{-jk_0 y \sin\theta} \\ E_{z_1} = \left(C_1 e^{jk_{x_1} x} + D_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta}. \end{cases} \tag{19}$$

From (3a), we can get

$$\begin{cases} H_{x_1} = \frac{k_0 \sin\theta}{\omega \mu_1} \left(C_1 e^{jk_{x_1} x} + D_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta} \\ H_{y_1} = \frac{k_{x_1}}{\omega \mu_1} \left(C_1 e^{jk_{x_1} x} - D_1 e^{-jk_{x_1} x} \right) e^{-jk_0 y \sin\theta}, \end{cases} \tag{20}$$

where $\vec{H_1} = H_{x_1} \vec{a_x} + H_{y_1} \vec{a_y}$ is the H field in dielectric and $\vec{H_0} = H_{x_0} \vec{a_x} + H_{y_0} \vec{a_y}$ is for air by replacing all subscript 1's with 0's.

Since $E_{z_1}|_{x=0_+} = 0$ and $E_{z_0}|_{x=d_+} = E_{z_1}|_{x=d_-}$, we have $D_1 = -C_1$ and, with $R_{\perp} = D_0/C_0$,

$$C_1 = \frac{C_0 \left(e^{jk_{x_0} d} + R_{\perp} e^{-jk_{x_0} d} \right)}{e^{jk_{x_1} d} - e^{-jk_{x_1} d}}. \tag{21}$$

Another boundary condition is $H_{y_0}|_{x=d_+} = H_{y_1}|_{x=d_-}$, which is

$$\begin{aligned} &\frac{k_{x_0} C_0}{\mu_0} \left(e^{jk_{x_0} d} - R_{\perp} e^{-jk_{x_0} d} \right) \\ =&\frac{k_{x_1} C_1}{\mu_1} \left(e^{jk_{x_1} d} + e^{-jk_{x_1} d} \right). \end{aligned} \tag{22}$$

Replacing (21) into (22), we can solve

$$R_{\perp} = \frac{j\mu_1 k_{x_0} \sin k_{x_1} d - \mu_0 k_{x_1} \cos k_{x_1} d}{j\mu_1 k_{x_0} \sin k_{x_1} d + \mu_0 k_{x_1} \cos k_{x_1} d} e^{2jk_{x_0} d}. \tag{23}$$

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